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Imaginary elements in geometry according to Von Staudt and Klein

VERLAG AM GOETHEANUM

For Johanna,

with love and gratitude

Preface

A remarkable fact in mathematics is the accordance between algebra and geometry: since the time of Descartes it is possible to express geometric phenomena in terms of numbers. And after doing calculations with these numbers, we can draw geometric conclusions from them. However, soon it appeared that, for instance, a circle and a line outside that circle have 'imaginary' meeting points: points that have imaginary coordinates but can *not* be found in the figure.

By extending each line to a Gauss- or Argand-plane, there is a possibility to add geometric meaning to these points. The big advantage again is the power of complex arithmetic that becomes available. But for the geometric imagination, complex geometry is very difficult if not impossible. And above all: not very satisfactory.

Karl von Staudt found a brilliant way to visualize imaginary geometric elements, in the real plane as well as in space (see [Staudt1860]). Felix Klein simplified his method ([Klein1872], which is reproduced in chapter 15 of the appendix). In our book we elaborate on the Klein-method. But it should be noted beforehand that the methods of Von Staudt and Klein cannot be thought of as the ultimate idea of imaginary elements either (see [Boer2012]). The reason that I took the trouble to write about this subject is not so much to add to the mathematical knowledge as well as to propose an alternative for the remarkable occurrence of complex numbers in various branches of *physics*. I recommend physicists to take notice of section 3.3 where the essential application to physics is presented.

In part I we develop 1-dimensional geometry of complex elements, in the second part dimension 2 is treated, and in the third part dimension 3. Mathematicians with sufficient background can skip the first two parts. Much emphasis is on the connection between the Klein-space and the numerical one.

The reader is supposed to be familiar with elementary projective geometry and linear algebra. Good guides are [Lipschutz1991] for algebra and [Ayres1967] for projective geometry, or [Baer2005] for both.

There are several ways to develop projective geometry. The easiest way is no doubt the numerical one which is summarized in chapters 6 and 10. It is, however, possible to develop projective geometry by synthetic geometric means. Namely the vector space can be constructed by a pure synthetic geometric procedure. The first attempt to this was done by Von Staudt, see [Staudt1860]. To my knowledge Artin was the first to give a full treatment of this construction, see [Artin1957]. In [Boer2009] you find a complete synthetic axiom system, as well as the Artin-construction of the vector space.

This book has evolved from a series of lectures I gave on the subject to a group of colleagues in the years 2016-2018. I am truly honoured and grateful for their patience and support. In particular Matthias Lerchmüller, who simultaneously treated the same subject from the point of view of Van Staudt, inspired me and gave essential help. Bernard Asselbergs took the time to carefully read parts 1 and 2 of the manuscript and gave many corrections and hints for improvement. Thanks a lot Bernard! And last but not least my dear wife, Johanna, who is always there, loving and supporting: to her I dedicate this work.

Lou de Boer, autumn 2020

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