

L.A.D. de Boer

**Imaginary elements in geometry
according to Von Staudt and Klein**

VERLAG AM GOETHEANUM

*For Johanna,
with love and gratitude*

Preface

A remarkable fact in mathematics is the accordance between algebra and geometry: since the time of Descartes it is possible to express geometric phenomena in terms of numbers. And after doing calculations with these numbers, we can draw geometric conclusions from them. However, soon it appeared that, for instance, a circle and a line outside that circle have ‘imaginary’ meeting points: points that have imaginary coordinates but can *not* be found in the figure.

By extending each line to a Gauss- or Argand-plane, there is a possibility to add geometric meaning to these points. The big advantage again is the power of complex arithmetic that becomes available. But for the geometric imagination, complex geometry is very difficult if not impossible. And above all: not very satisfactory.

Karl von Staudt found a brilliant way to visualize imaginary geometric elements, in the real plane as well as in space (see [Staudt1860]). Felix Klein simplified his method ([Klein1872], which is reproduced in chapter 15 of the appendix). In our book we elaborate on the Klein-method. But it should be noted beforehand that the methods of Von Staudt and Klein cannot be thought of as the ultimate idea of imaginary elements either (see [Boer2012]). The reason that I took the trouble to write about this subject is not so much to add to the mathematical knowledge as well as to propose an alternative for the remarkable occurrence of complex numbers in various branches of *physics*. I recommend physicists to take notice of section 3.3 where the essential application to physics is presented.

In part I we develop 1-dimensional geometry of complex elements, in

the second part dimension 2 is treated, and in the third part dimension 3. Mathematicians with sufficient background can skip the first two parts. Much emphasis is on the connection between the Klein-space and the numerical one.

The reader is supposed to be familiar with elementary projective geometry and linear algebra. Good guides are [Lipschutz1991] for algebra and [Ayres1967] for projective geometry, or [Baer2005] for both.

There are several ways to develop projective geometry. The easiest way is no doubt the numerical one which is summarized in chapters 6 and 10. It is, however, possible to develop projective geometry by synthetic geometric means. Namely the vector space can be constructed by a pure synthetic geometric procedure. The first attempt to this was done by Von Staudt, see [Staudt1860]. To my knowledge Artin was the first to give a full treatment of this construction, see [Artin1957]. In [Boer2009] you find a complete synthetic axiom system, as well as the Artin-construction of the vector space.

This book has evolved from a series of lectures I gave on the subject to a group of colleagues in the years 2016-2018. I am truly honoured and grateful for their patience and support. In particular Matthias Lerchmüller, who simultaneously treated the same subject from the point of view of Van Staudt, inspired me and gave essential help. Bernard Asselbergs took the time to carefully read parts 1 and 2 of the manuscript and gave many corrections and hints for improvement. Thanks a lot Bernard! And last but not least my dear wife, Johanna, who is always there, loving and supporting: to her I dedicate this work.

Lou de Boer, autumn 2020

Contents

I	1-dimensional geometry	11
1	The theory of Von Staudt	13
1.1	The common points of conic and line	13
1.2	Orientation of the line	15
1.3	Imaginary points	15
1.4	Coordinates	16
2	The theory of Klein	19
2.1	Definition	19
2.2	Coordinates	20
2.3	Klein versus Von Staudt	23
3	Groups of automorphisms	25
3.1	Elliptic maps	25
3.2	Drawback	28
3.3	Application to physics	28
4	Constructions	31
4.1	The fundamental construction	31
4.2	The dual construction	33
4.3	The fundamental construction in space	35
4.4	Construction of an imaginary point	36
4.5	Construction of an imaginary line	38
4.6	Construction of an involution	39
4.7	Constructing a Klein-map with a conic	40
II	2-dimensional geometry	43
5	The Klein-plane	45
5.1	Imaginary elements	45
5.2	Ordering	46

5.3	The Staudt-plane	49
6	The numerical plane	51
7	The bijection	55
7.1	The coordinate map κ	55
7.2	Consistency	58
7.3	The inverse of κ	59
7.4	The invariance of \prec	61
7.5	More objections	62
8	Conic and line	65
8.1	Orientation on a conic	65
8.2	Von Staudt's method	66
8.3	Klein's method	68
8.4	An example	69
8.5	Conversion Klein-Staudt	70
9	Extension to the plane?	73
9.1	The cubic	73
9.2	W-curves	75
III	3-dimensional geometry	79
10	The numerical space	81
10.1	Definitions	81
10.2	Plücker-coordinates	82
10.3	Ordering, join, meet	84
10.4	Types of lines	86
10.5	Projective maps	87
10.6	The linear congruence	87
11	The Klein-space	93
11.1	The synthetic space	93
11.2	Low imaginary elements	94
11.3	The high imaginary line	94
11.4	The matrix of a high imaginary line	97
11.5	Sets and numbers	99
11.6	Ordering	100
11.7	Meet and join	104
11.8	Is our Klein-space a projective one?	108

12 The bijection for 3-d **111**

 12.1 Point and plane coordinates 111

 12.2 The coordinates of a line 113

 12.3 The inverse of κ 116

13 Invariance of dim and \prec **119**

 13.1 Dimension 119

 13.2 Containment 120

14 W-curves in space **123**

IV Appendix **127**

15 Klein’s original text **129**

16 Projective spaces **133**

 16.1 Definition of ‘projective space’ 133

 16.2 Subspaces 135

 16.3 Isomorphic spaces 135

 16.4 Eigenspaces 137

17 Geometry of the line **139**

 17.1 Orientation 139

 17.2 Separation 139

 17.3 Cross Ratio 140

 17.4 Maps of the real line 140

 17.5 Maps of the complex line 142

 17.6 Splitting matrices 143

 17.7 The standard elliptic map 145

18 The Klein-arrow **147**

 18.1 Arrow and twist in the plane 147

 18.2 Extension to space 151

19 Various **155**

 19.1 Non-integer powers of a matrix 155

 19.2 An image of the linear congruence 156

 19.3 Postponed proofs 160

 19.4 Three non-concurrent lines in the plane 163

 19.5 The projective image of a line 164

Bibliography **167**

List of symbols	169
Index	171